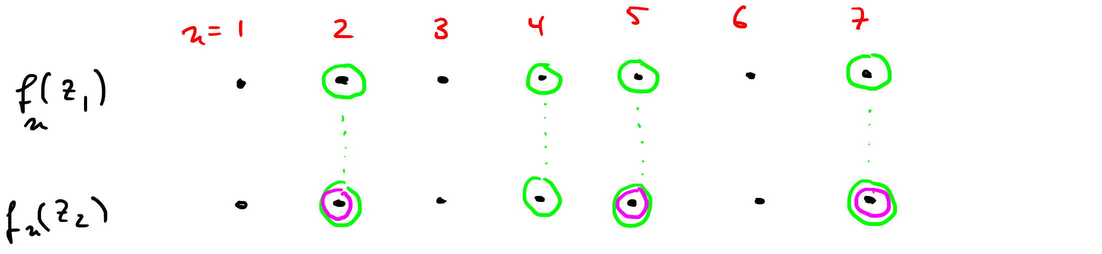
Topics in complex analysis. Lecture 2 (Sept 23, 2024)

Last time
- Notions of cvg of complex fcts (ptw, unif, loc unif)
- which props are stable?

Pf of Thm 1.7 Step 1 Def the limit fct at ctbly many points
Compactness

Step 2 "Complete" the limit fct beyond this set
Equicontinuity

Take $S \subseteq U$ countable & dense $S = \{z_1, z_2, \dots\}$



"Diagonal procedure"

In the j'th step we have chosen a subsequ $(n_{k,j})_{k\in\mathbb{N}}$
st. $f_{n_{k,j}}(z_j) \longrightarrow \exists f_{z_j} \in \mathbb{C}$ For $k \in \mathbb{N}$ set $\boxed{n_k = n_{k,k}}$

Then $(f_{n_k}(z_j))_{k\in\mathbb{N}}$ conv. to f_{z_j} $\forall j \in \mathbb{N}$.

Compactness

Claim (f_n) equicontinuous, ie.

$$\forall \varepsilon>0 \;\; \exists \delta>0 \;\; \forall z \in \mathcal{U}: \quad |z-z'| \leq \delta$$

$$\Longrightarrow \sup_{n\in\mathbb{N}} \underbrace{|f_n(z)-f_n(z')|}_{\leq L|z-z'|^\alpha} \leq \varepsilon$$

Let $z_0 \in \mathcal{U}$ and $r>0$ st (a) $\overline{B}_{2r}(z_0) \subseteq \mathcal{U}$

(b) $\sup_{n\in\mathbb{N}} \sup_{z\in\overline{B}_{2r}(z_0)} |f_n(z)| < \infty$

Cauchy integral formula yields $\forall z' \in B_r(z_0)$

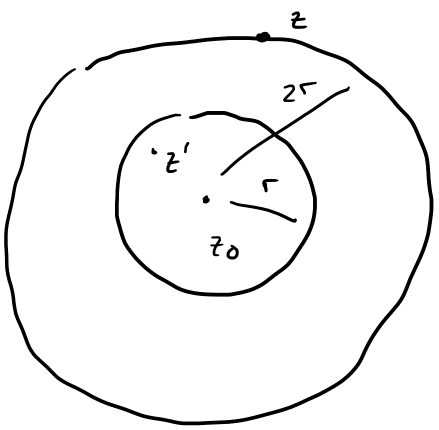
$|f_n(z')-f_n(z_0)|$

$$= \left| \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f_n(z)}{z-z'} - \frac{f_n(z)}{z-z_0} \, dz \right|$$

$$= \left| \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{f_n(z)(z-z_0) - f_n(z)(z-z')}{(z-z')(z-z_0)} \, dz \right|$$

good!

$$\leq \frac{1}{2\pi} |z_0-z'| \int_{\partial B_{2r}(z_0)} \frac{\overbrace{|f_n(z)|}^{\leq C}}{\underbrace{|z-z'|}_{\geq r}\,\underbrace{|z-z_0|}_{=2r}} \, dz$$



$$\leq \frac{C}{r} |z'-z_0|.$$

Choosing $\delta_\varepsilon = \min\{r, \varepsilon \frac{r}{C}\}$

yields the desired equicty.

Hence (f_n) equicts (Hence $(f_{n_k}(z))_k$ Cauchy $\forall z \in U$, hence $\exists$ limit $f_z \in \mathbb{C}$)

Given $\varepsilon > 0$ & $z \in U$ choose $z^* \in S$ st $|z - z^*| \leq \delta_{\varepsilon, z}$ where $\delta_{\varepsilon z}$ satisfies

$$|y - z| \leq \delta_{\varepsilon z} \Rightarrow \sup_n |f_n(y) - f_n(z)| \leq \varepsilon.$$

Then $\forall k \geq \ell$

$$\boxed{|f_{n_k}(z) - f_{n_\ell}(z)|}$$

$$\leq \underbrace{|f_{n_k}(z) - f_{n_k}(z^*)|}_{\leq \varepsilon} + \overbrace{|f_{n_k}(z^*) - f_{n_\ell}(z^*)|}^{\text{converges!}}$$

$$+ \underbrace{|f_{n_\ell}(z) - f_{n_\ell}(z^*)|}_{\leq \varepsilon}$$

$$\leq 2\varepsilon + |f_{n_k}(z^*) - f_{n_\ell}(z^*)|$$

Since $(f_{n_k}(z^*))_{k \in \mathbb{N}}$ Cauchy by Step 1 $\forall z^* \in S$ the rhs becomes $\leq 3\varepsilon$ for k, ℓ large.

Now set $f(z) := f_z$ for $z \in U$. Then f_{n_k} converges pointwise to f. This can be upgraded to loc unif cvg using equicty. $\square$

Lemma (a_n) sequence of complex numbers, $a \in \mathbb{C}$.
Then $a_n \to a$ iff every subsequ of (a_n) has a further subsequ converging to a.

Proof "$\Rightarrow$" trivial

"$\Leftarrow$" Assume $(a_n)_{n \in \mathbb{N}}$ doesn't cvg to a. Then

$$\exists \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists n_N \geq N : \quad |a_{n_N} - a| \geq \varepsilon$$

The subsequ $(a_{n_N})_{N \in \mathbb{N}}$ thus constructed has no subsequ which cvg to a. $\square$

Pf of Thm 1.8. Montel's thm $\Rightarrow \exists (f_{n_k})_{k \in \mathbb{N}}$ loc unif to some holomorphic fct $f : D \to \mathbb{C}$.

On the other hand assume (f_n) doesn't cvg to f. Then the lemma implies $\exists (f_{n_\ell})_{\ell \in \mathbb{N}}$ st no further subsequ of it cvg loc unif to f.
Applying Montel to $(f_{n_\ell})_{\ell \in \mathbb{N}}$ yields loc unif cvg of a subsequence to some $h \neq f$.

But $h(z) = f(z) \quad \forall z \in L$ Thm 0.6

But L has accumulation pt in $D \Rightarrow h = f$ $\lightning$